# Some discrete integrable equations related to an elliptic curve

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## Contents



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[1] V.E. Adler, Yu.B. Suris. Q4: Integrable Master Equation Related to an Elliptic Curve. *Int. Math. Res. Not.* **47** (2004) 2523–2553.

# 1 Introduction

We discuss several 2D integrable systems with the spectral parameter on an elliptic curve: Landau-Lifshitz (LL) and Krichever-Novikov (KN) equations, elliptic Toda lattice (ETL) and elliptic Ruijsenaars-Toda lattice (ERTL). These models can be unified on the basis of a single discrete equation (subscripts denote the shifts in the square lattice):

$$(\mathrm{Q}_4) \qquad abc(uu_1u_2u_{12}+1)+a(uu_1+u_2u_{12})-b(uu_2+u_1u_{12})-c(uu_{12}+u_1u_2)=0$$

$$A^2=a^4-da^2+1,\quad B^2=b^4-db^2+1,\quad c=\frac{aB-Ab}{1-a^2b^2}.$$

| | | | | |
|---|---|---|---|---|
| $\Gamma$ | ($\mathrm{Q}_4$) on quad-graphs | $\longrightarrow$ | discrete ETL on graphs | |
| | $\downarrow$ | | $\swarrow$ | $\searrow$ |
| $\mathbb{Z}^2$ | nonlinear superposition | | square lattice | triangular lattice |
| | $\downarrow$ | | $\downarrow$ | $\downarrow$ |
| $\mathbb{Z}\times\mathbb{R}$ | Bäcklund transformation | | ETL | ERTL |
| | $\downarrow$ | | $\searrow$ | $\swarrow$ |
| $\mathbb{R}^2$ | KN | | LL | |

Remark. Eq. ($Q_4$) was introduced in [2] and studied further in [3, 4, 1]. However, in these papers it was presented in much more cumbersome form related to the Weierstrass form of the elliptic curve ($A^2 = 4a^3 - g_2 a - g_3$). The given form of equation (Q4) was found by Hietarinta [SIDE-2004 talk].

[2] V.E. Adler. Bäcklund transformation for the Krichever-Novikov equation. *Int. Math. Res. Notices* **1998:1** 1–4.

[3] V.E. Adler, A.I. Bobenko, Yu.B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. *Comm. Math. Phys.* **233** (2003) 513–543.

[4] F. Nijhoff. Lax pair for the Adler (lattice Krichever-Novikov) system. *Phys. Lett. A* **297** (2002) 49–58.

# 2 Nonlinear superposition for the Krichever-Novikov equation

The Krichever-Novikov equation [5]

(1) $$u_t = u_{xxx} - \frac{3(u_{xx}^2 - r(u))}{2u_x}, \quad r^{(5)} = 0$$

is the most generic nonlinear integrable equation of the form $u_t = u_{xxx} + f(u_{xx}, u_x, u)$. Accordingly to [6], all other equations of this type are related via differential substitutions to eq (1) when some zeroes of the polynomial $r$ are multiple, while the case of simple zeroes is isolated.

Bäcklund auto-transformation for (1) is of the form [2]

(2) $$u_x \tilde{u}_x = h(u, \tilde{u})$$

where $h$ is the biquadratic polynomial in $u, \tilde{u}$, such that

$$r(u) = h_{\tilde{u}}^2 - 2hh_{\tilde{u}\tilde{u}}, \quad r(\tilde{u}) = h_u^2 - 2hh_{uu}.$$

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[5] I.M. Krichever, S.P. Novikov. Holomorphic bundles over algebraic curves and nonlinear equations. *Uspekhi Mat. Nauk* **35:6** (1980) 47–68.

[6] S.I. Svinolupov, V.V. Sokolov, R.I. Yamilov. Bäcklund transformations for integrable evolution equations. *Dokl. Akad. Nauk SSSR* **271:4** (1983) 802–805.

The polynomial $h$ corresponding to $r(u) = u^4 - du^2 + 1$ depends on an additional parameter $(a, A)$ on the elliptic curve $A^2 = r(a)$:

$$h(u, \tilde{u}; a, A) = \frac{1}{2a}(a^2u^2\tilde{u}^2 - 2Au\tilde{u} - u^2 - \tilde{u}^2 + a^2).$$

BTs corresponding to the different values of $\alpha$ commute, and eq. ($Q_4$) defines the nonlinear superposition principle for these BT. This means the following. The equations

$$\begin{aligned} u_x u_{1,x} &= h(u, u_1; a, A) \qquad & u_{2,x}u_{12,x} &= h(u_2, u_{12}; a, A) \\ u_x u_{2,x} &= h(u, u_2; b, B) \qquad & u_{1,x}u_{12,x} &= h(u_1, u_{12}; b, B) \end{aligned} \tag{3}$$

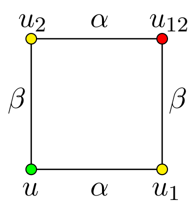
imply the reducible constraint

$$h(u, u_1; a, A)h(u_2, u_{12}; a, A) - h(u, u_2; b, B)h(u_1, u_{12}; b, B) = Q\tilde{Q} = 0$$

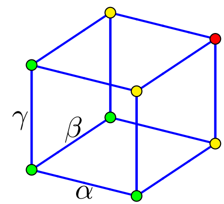
where $Q$ is the l.h.s. of the eq. ($Q_4$) and $\tilde{Q} = Q|_{b\to -b}$. The statement is that the constraint $Q = 0$ is consistent with the dynamics on $x$ defined by (3): $\left.\frac{dQ}{dx}\right|_{Q=0} = 0$.

The important property of ($Q_4$) eq. is $3D$-consistency, or consistency around a cube. This property means that if we assign 6 copies of ($Q_4$) to the faces of a cube, with the common values of the parameters on the edges, then for arbitrary choice of initial data $u, u_1, u_2, u_3$ the values $u_{123}$ calculated in three posssible ways coincide.

The classification of $3D$-consistent equation, under some additional assumptions, was obtained in [9].




nonlinear superposition




$3D$-consistency


[7] F.W. Nijhoff, A.J. Walker. The discrete and continuous Painlevé hierarchy and the Garnier system. *Glasgow Math. J.* **43A** (2001) 109–123.

[8] A.I. Bobenko, Yu.B. Suris. Integrable systems on quad-graphs. *Int. Math. Res. Notices* **11** (2002) 573–611.

[9] V.E. Adler, A.I. Bobenko, Yu.B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. *Comm. Math. Phys.* **233** (2003) 513–543.

# 3 Three-leg form of ($Q_4$)

Statement 1. Equation ($Q_4$) is equivalent, under the changes

$$a=\delta^{-1}\operatorname{sn}\alpha,\ A=\operatorname{sn}'\alpha,\quad b=\delta^{-1}\operatorname{sn}\beta,\ B=\operatorname{sn}'\beta,\quad c=\delta^{-1}\operatorname{sn}(\alpha-\beta),\quad u=\delta^{-1}\operatorname{sn}q$$

(where $\operatorname{sn}x\equiv\operatorname{sn}(x,d^{-2})$), to the equation

$$F(q,q_1,\alpha)/F(q,q_2,\beta)=F(q,q_{12},\alpha-\beta) \tag{4}$$

where

$$F(q,\widetilde{q},\alpha)=\frac{\operatorname{sn}(q+\alpha)-\operatorname{sn}(\widetilde{q})}{\operatorname{sn}(q-\alpha)-\operatorname{sn}(\widetilde{q})}\cdot\frac{\Theta_4(q+\alpha)}{\Theta_4(q-\alpha)}$$

This property implies several important consequences: $S_4$ symmetry group of the equation, $3D$-consistency, relation to discrete Toda type lattices.

# 4 $S_4$ symmetry group of ($Q_4$)

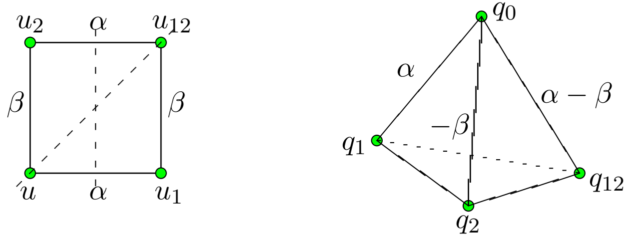
Obviously, eq ($Q_4$) admits the symmetry group $D_4$ of the square:

$$Q(u, u_1, u_2, u_{12}, \alpha, \beta) = Q(u_1, u, u_{12}, u_2, \alpha, \beta) = -Q(u, u_2, u_1, u_{12}, \beta, \alpha).$$

Due to this symmetry the three-leg form can be centered in an arbitrary vertex. On the other hand, three-leg form exhibits one more symmetry which is hidden in the rational form: the diagonals of the quadrilateral are on the equal footing with its edges.

Indeed, due to the property $F(q, \widetilde{q}, \alpha) = 1/F(q, \widetilde{q}, -\alpha)$, eq (4) can be rewritten in the form

$$F(q, q_1, \alpha)F(q, q_2, -\beta)F(q, q_{12}, \beta - \alpha) = 1.$$



# 5 Three-leg form implies $3D$-consistency

Theorem 2. Eq ($Q_4$) is $3D$-consistent.

*Proof.* Consider the system of equations, associated to the faces of the cube:

$$
\begin{array}{llll}
(E_{12}) & Q(u,u_1,u_2,u_{12},\alpha_1,\alpha_2)=0 & (\widetilde{E}_{12}) & Q(u_3,u_{13},u_{23},u_{123},\alpha_1,\alpha_2)=0\\
(E_{13}) & Q(u,u_1,u_3,u_{13},\alpha_1,\alpha_3)=0 & (\widetilde{E}_{13}) & Q(u_2,u_{13},u_{23},u_{123},\alpha_1,\alpha_3)=0\\
(E_{23}) & Q(u,u_2,u_3,u_{23},\alpha_2,\alpha_3)=0 & (\widetilde{E}_{23}) & Q(u_1,u_{12},u_{13},u_{123},\alpha_2,\alpha_3)=0
\end{array}
$$

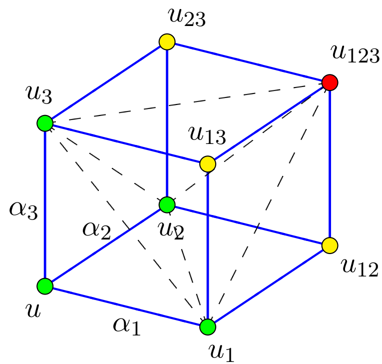
One have to prove that if the values $u_{12},u_{13},u_{23}$ are defined from eqs on the left, for arbitrary initial data $u,u_1,u_2,u_3$, then the rest eqs define one and the same value $u_{123}$.

It is enough to show that if $u_{123}$ is defined by eq ($\widetilde{E}_{23}$), then ($\widetilde{E}_{13}$) is fulfilled as well. Rewrite eqs, containing $u_1$, in three-leg forms:

$$
\begin{gathered}
F(q_1,q_{13},\alpha_3)/F(q_1,q,\alpha_1)=F(q_1,q_3,\alpha_3-\alpha_1),\\
F(q_1,q_{12},\alpha_2)/F(q_1,q,\alpha_1)=F(q_1,q_2,\alpha_2-\alpha_1),\\
F(q_1,q_{12},\alpha_2)/F(q_1,q_{13},\alpha_3)=F(q_1,q_{123},\alpha_2-\alpha_3).
\end{gathered}
$$

From here the equation

$$F(q_1,q_2,\alpha_2-\alpha_1)/F(q_1,q_3,\alpha_3-\alpha_1)=F(q_1,q_{123},\alpha_2-\alpha_3)$$




follows, relating the fields at the vertices of the dashed tetrahedron. This is nothing but the three-leg form of the equation

$$Q(u_1, u_2, u_3, u_{123}, \alpha_2 - \alpha_1, \alpha_2 - \alpha_3) = 0,$$

centered at $q_1$. This can be centered at $q_2$, as well, resulting in the cyclic shift of indices:

$$F(q_2, q_3, \alpha_3 - \alpha_2)/F(q_2, q_1, \alpha_1 - \alpha_2) = F(q_2, q_{123}, \alpha_3 - \alpha_1).$$

The latter equation, together with the three-leg forms of equations $(E_{12})$, $(E_{23})$ centered at $q_2$, leads to the three-leg form of $(\widetilde{E}_{13})$, as required. □

# 6 $3D$-consistency → zero curvature representation

An affine-linear equation $Q=0$ may be interpreted as Möbius transformation between any pair of variables, with coefficients depending on the rest pair. Let

$$u_{13}=M(u_1,u,\alpha_1,\alpha_3)[u_3]=\frac{Au_3+B}{Cu_3+D}$$

then

$$u_{23}=M(u_2,u,\alpha_2,\alpha_3)[u_3],\quad u_{123}=M(u_{12},u_2,\alpha_1,\alpha_3)[u_{23}]=M(u_{12},u_1,\alpha_2,\alpha_3)[u_{13}].$$

Since the composition of Möbius transformations corresponds to the product of the matrices, hence denoting $\alpha_3=\lambda$ and introducing the normalization factor yields the zero curvature representation

$$L(u_{12},u_1,\alpha_2,\lambda)L(u_1,u,\alpha_1,\lambda)=L(u_{12},u_2,\alpha_1,\lambda)L(u_2,u,\alpha_2,\lambda)$$

with the matrix

$$L(u_1,u,\alpha_1,\lambda)=(AD-BC)^{-1/2}\begin{pmatrix}A & B\\ C & D\end{pmatrix}.$$

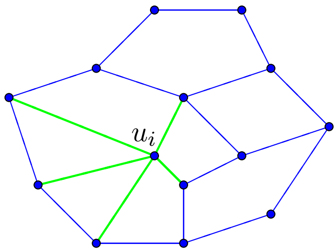
For the ($Q_4$) equation one obtains ($a=d^{-1}\operatorname{sn}\alpha$, $l=d^{-1}\operatorname{sn}\lambda$, $m=d^{-1}\operatorname{sn}(\alpha-\lambda)$)

$$L(u_1,u,\alpha_1,\lambda)=h(u,u_1,\alpha)^{-1/2}\begin{pmatrix}lu+mu_1 & -alm-auu_1\\ almuu_1+a & -lu_1-mu\end{pmatrix}.$$

# 7 Discrete Toda lattices

Discrete Toda lattices can be defined as equations on "stars" for arbitrary planar graph $G$:

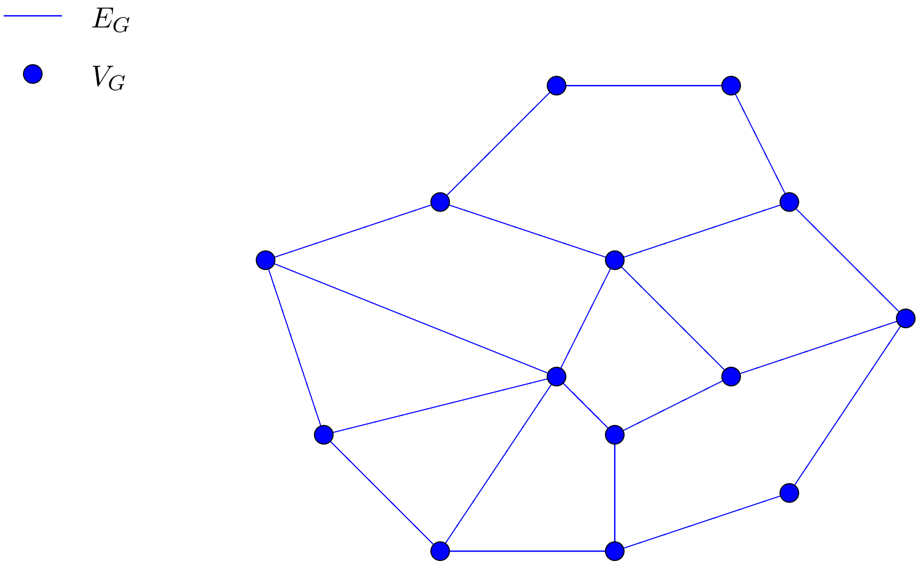
$$\sum_{j:(i,j)\in E_G} f_{ij}(u_i, u_j) = 0$$

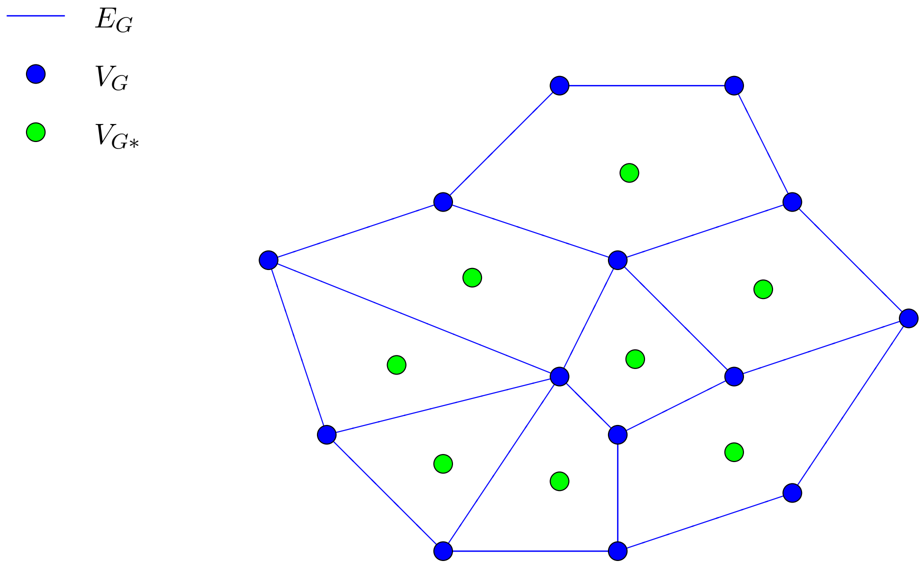


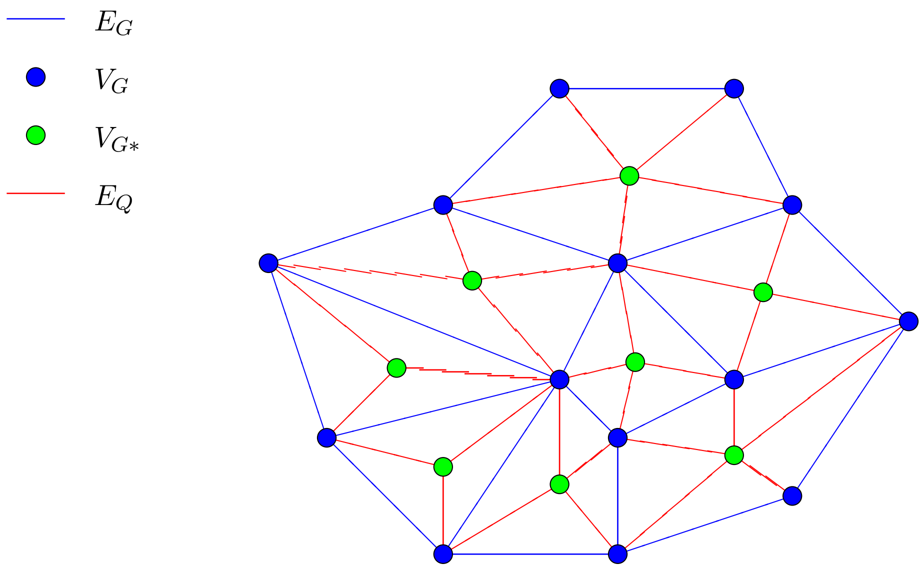

There exists the correspondence between the planar graphs $G$ and the bipartite quad-graphs $Q$:

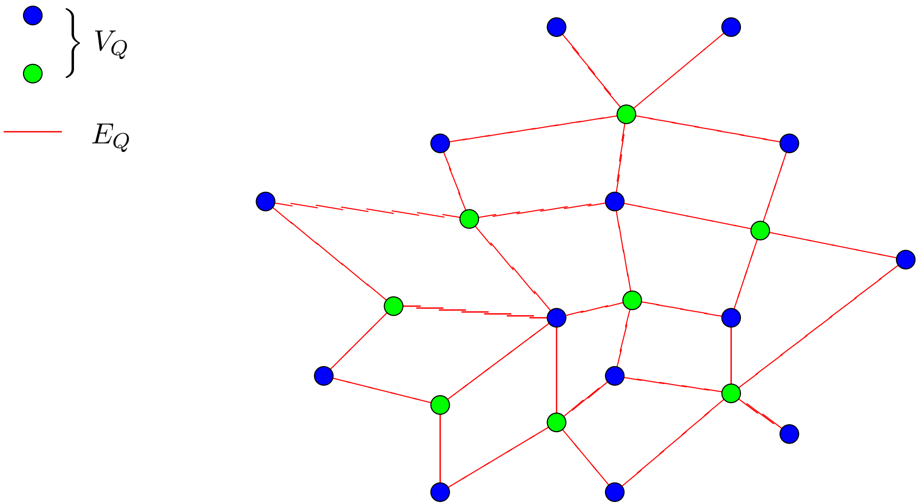
$$V_Q = V_G \cup V_{G^*}, \qquad E_Q = \{(i, i^*) |\ i \in V_G,\ i^* \in V_{G^*},\ i \in f(i^*)\}$$

where $f(i^*)$ is the face of $G$, corresponding to the vertex $i^*$ of dual graph. In other words, the edges of $G$ are diagonals of the faces of $Q$ joining the vertices of one of two types.


$E_G$
$V_G$


$E_G$
$V_G$
$V_{G*}$

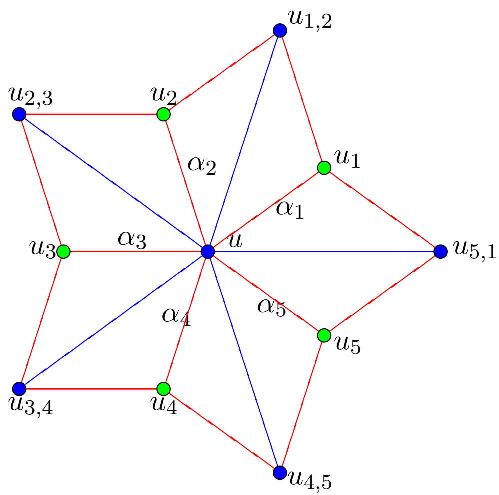

$E_G$
$V_G$
$V_{G*}$
$E_Q$


$V_Q$
$E_Q$

Now, associate eq ($Q_4$) to each face of a quad-graph. In all faces, incident to some blue vertex, consider the three-leg form of equation centered in this vertex. Then the product of these equations is free from the variables associated to the green vertices:

$$\prod_{j=1}^{n} F(q, q_{j,j+1}, \alpha_{j+1} - \alpha_j) = 1 \tag{5}$$

In the original variables $u$ this equation is rational.



The eliminated variables $u_j$ play the role of $\psi$-functions: their telescopic cancellation means that the composition of the Möbius transformations

$$u_2 = M(u, u_{1,2}, \alpha_1, \alpha_2)[u_1], \quad u_3 = M(u, u_{2,3}, \alpha_2, \alpha_3)[u_2], \quad \dots \\ u_n = M(u, u_{n-1,n}, \alpha_{n-1}, \alpha_n)[u_{n-1}], \quad u_1 = M(u, u_{n,1}, \alpha_n, \alpha_1)[u_n]$$

is the identity transformation. In the matrix language,

$$\overset{\curvearrowleft}{\prod_{1\le j\le n}} L(u, u_{j,j+1}, \alpha_j + \lambda, \alpha_{j+1} + \lambda) = \mathrm{const}\, I$$

where $\lambda$ appears due to the shift invariance of (5).

Remark. The $3D$-consistent equations were classified in [3], under some additional assumptions. The legs for the other equations of the list can be obtained as limiting cases:

| | $F(q,\widetilde{q},\alpha)$ | $u=u(q)$ | $a=a(\alpha)$ |
|---|---|---|---|
| $(Q_1)_{\delta=0}$ | $\exp(\alpha/(q-\widetilde{q}))$ | $q$ | $\alpha$ |
| $(Q_1)_{\delta=1}$ | $\dfrac{q-\widetilde{q}+\alpha}{q-\widetilde{q}-\alpha}$ | $q$ | $\alpha$ |
| $(Q_2)$ | $\dfrac{(q+\widetilde{q}+\alpha)(q-\widetilde{q}+\alpha)}{(q+\widetilde{q}-\alpha)(q-\widetilde{q}-\alpha)}$ | $q^2$ | $\alpha$ |
| $(Q_3)_{\delta=0}$ | $\dfrac{\sinh(q-\widetilde{q}+\alpha)}{\sinh(q-\widetilde{q}-\alpha)}$ | $\exp 2q$ | $\exp 2\alpha$ |
| $(Q_3)_{\delta=1}$ | $\dfrac{\sinh(q+\widetilde{q}+\alpha)\sinh(q-\widetilde{q}+\alpha)}{\sinh(q+\widetilde{q}-\alpha)\sinh(q-\widetilde{q}-\alpha)}$ | $\cosh 2q$ | $\exp 2\alpha$ |

The leg for the version of $(Q_4)$ corresponding to the ellipic curve in Weierstrass form:

(6)
$$F(q,\widetilde{q},\alpha)=\frac{\sigma(q+\widetilde{q}+\alpha)\sigma(q-\widetilde{q}+\alpha)}{\sigma(q+\widetilde{q}-\alpha)\sigma(q-\widetilde{q}-\alpha)}.$$

# 8 Elliptic Toda lattice

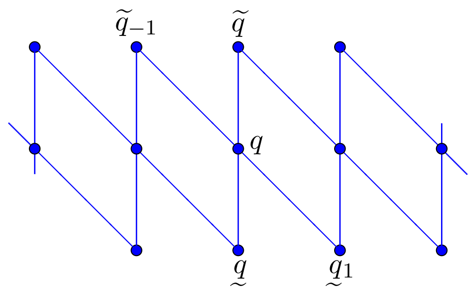
Cosider the discrete Toda system on the skew square lattice:

$$F(q,\widetilde{q},\varepsilon)F(q,\widetilde{q}_{-1},-\varepsilon)F(q,\underset{\sim}{q},\varepsilon)F(q,\underset{\sim}{q}_1,-\varepsilon)=1$$

([10], in the rational variables $u$). It can be written in the Hamiltonian form

$$p=-f(q,\widetilde{q},\varepsilon)+f(q,\widetilde{q}_{-1},\varepsilon),\quad \widetilde{p}=f(\widetilde{q},q,\varepsilon)-f(\widetilde{q},q_1,\varepsilon)$$

where $f=\log F$. Here and in the next section we will use the leg in the form (6).




[10] V.E. Adler. Discretizations of the Landau-Lifshitz equation. *Teor. Math. Phys.* **124:1** (2000) 897–908.

Consider the continuous limit $\widetilde{q} = q + \varepsilon q_x$. Taking into account the relations

$$\lim_{\varepsilon\to 0} f(\widetilde{q}, q, \varepsilon) = \log \frac{q_x + 1}{q_x - 1}, \quad \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(f(\widetilde{q}, q, \varepsilon) + f(q, \widetilde{q}, \varepsilon)) = 2\zeta(2q),$$

one finds

$$p = \log \frac{q_x + 1}{q_x - 1},$$
$$p_x = -\zeta(q + q_1) + \zeta(q_1 - q) - \zeta(q + q_{-1}) - \zeta(q - q_{-1}) + 2\zeta(2q).$$

From here the Newtonian equations follow:

$$\frac{q_{xx}}{q_x^2 - 1} = \zeta(q_1 + q) - \zeta(q_1 - q) + \zeta(q + q_{-1}) + \zeta(q - q_{-1}) - 2\zeta(2q).$$

This is the elliptic Toda lattice as given in [11]. The rational form of this equation [12, 13] reads

$$\frac{u_{xx} - r'(u)/2}{u_x^2 - r(u)} = \frac{1}{u - u_1} + \frac{1}{u - u_{-1}}.$$

[11] I.M. Krichever. Elliptic analog of the Toda lattice. *Int. Math. Res. Notices* **8** (2000) 383–412.
[12] A.B. Shabat, R.I. Yamilov. Symmetries of nonlinear chains, *Len. Math. J.* **2:2** (1991) 377.
[13] R.I. Yamilov. Classification of Toda type scalar lattices, Proc. NEEDS'93, World Scientific Publ., Singapore, 1993, 423–431.

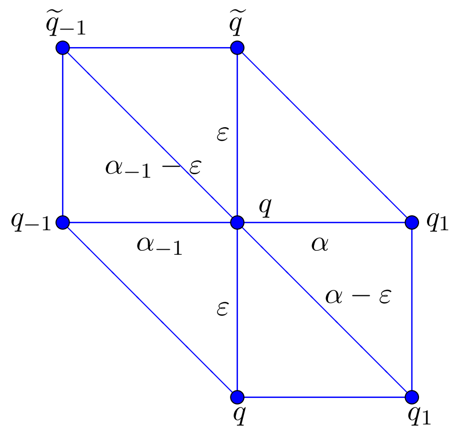
# 9 Elliptic Ruijsenaars-Toda lattice

Analogously, the Hamiltonian form of the discrete Toda system on the triangular lattice is

$$
\begin{aligned}
&p = -f(q, \widetilde{q}, \varepsilon) + f(q, q_1, \alpha) + f(q, q_{-1}, \alpha_{-1}) - f(q, \widetilde{q}_{-1}, \alpha_{-1} - \varepsilon),\\
&\widetilde{p} = f(\widetilde{q}, q, \varepsilon) + f(\widetilde{q}, q_1, \alpha - \varepsilon).
\end{aligned}
$$

Under the continuous limit one obtains

$$
\begin{aligned}
&p = \log \frac{q_x + 1}{q_x - 1} + f(q, q_1, \alpha),\\
&p_x = q_x f_q(q, q_1, \alpha) + q_{-1,x} f_{q_{-1}}(q, q_{-1}, \alpha_{-1}) - f_\alpha(q, q_1, \alpha) - f_{\alpha_{-1}}(q, q_{-1}, \alpha_{-1}) + 2\zeta(2q).
\end{aligned}
$$



From here the Newtonian equations of the elliptic Ruijsenaars-Toda lattice follow:

$$\frac{q_{xx}}{q_x^2-1}=q_{1,x}f_{q_1}(q,q_1,\alpha)-q_{-1,x}f_{q_{-1}}(q,q_{-1},\alpha_{-1})+f_\alpha(q,q_1,\alpha)+f_{\alpha_{-1}}(q,q_{-1},\alpha_{-1})-2\zeta(2q)$$

where

$$2f_{q_1}(q,q_1,\alpha)=\zeta(q+q_1+\alpha)-\zeta(q-q_1+\alpha)-\zeta(q+q_1-\alpha)+\zeta(q-q_1-\alpha),$$
$$2f_\alpha(q,q_1,\alpha)=\zeta(q+q_1+\alpha)+\zeta(q-q_1+\alpha)+\zeta(q+q_1-\alpha)+\zeta(q-q_1-\alpha).$$

The rational form of this lattice [14] reads

$$\frac{2u_{xx}-r'(u)}{u_x^2-r(u)}=-\frac{u_{1,x}}{h(u,u_1,\alpha)}+\frac{u_{-1,x}}{h(u,u_{-1},\alpha_{-1})}+\frac{\partial}{\partial u}\log\big(h(u,u_1,\alpha)h(u,u_{-1},\alpha_{-1})\big)$$

where $h(u,v,\alpha)$ is the symmetric biquadratic polynomial in $u,v$ such that $r(u)=h_v^2-2hh_{vv}$.

In the Hamiltonian form it appeared earlier in [12]

$$u_x=\frac{2h}{u_1-v}+h_v,\quad v_x=\frac{2h}{u-v_{-1}}-h_u,\quad h=h(u,v,\alpha)$$

[14] V.E. Adler, A.B. Shabat. On a class of Toda chains. *Theor. Math. Phys.* **111:3** (1997) 647–657.

and is closely related also to the elliptic Volterra lattice [15]

$$u_x = \frac{h(u_1, u, \alpha)}{u_1 - u_{-1}} - h_{u_1}(u_1, u, \alpha).$$

and Sklyanin lattice [16, 10].

[15] R.I. Yamilov. On classification of discrete evolution equations. *Usp. Mat. Nauk* **38:6** (1983) 155–156.

[16] E.K. Sklyanin. On some algebraic structures related to Yang-Baxter equation. *Funkts. analiz* **16:4** (1982) 27–34.